HOMOTOPY GERSTENHABER ALGEBRAS AND TOPOLOGICAL FIELD THEORY

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Abstract. We prove that the BRST complex of a topological conformal field theory is a homotopy Gerstenhaber algebra, as conjectured by Lian and Zuckerman in 1992. We also suggest a refinement of the original conjecture for topological vertex operator algebras. We illustrate the usefulness of our main tools, operads and "string vertices" by obtaining new results on Vassiliev invariants of knots and double loop spaces.

Two-dimensional topological quantum field theory (TQFT) at its most elementary level is the theory of $\mathbb{Z}$-graded commutative associative algebras (with some additional structure) [34]. Thus, it came as something of a surprise when several groups of mathematicians realized that the physical state space of a 2D TQFT has the structure of a $\mathbb{Z}$-graded Lie algebra, relative to a new grading equal to the old grading minus one. Moreover, the commutative and Lie products fit together nicely to give the structure of a Gerstenhaber algebra (G-algebra), a $\mathbb{Z}$-graded Poisson algebra for which the Poisson bracket has degree -1 (see Section 1). This G-algebra structure is best understood in the framework of 2D topological conformal field theories (TCFTs) (see Section 5.2) wherein operads of moduli spaces of Riemann surfaces play a fundamental role.

G-algebras arose explicitly in M. Gerstenhaber's work on the Hochschild cohomology theory for associative algebras (see Section 1 for this and several other contexts for the theory of G-algebras). Operads arose in the work of J. Stasheff, Gerstenhaber and later work of P. May on the recognition problem for iterated loop spaces. Eventually, F. Cohen discovered that the homology of a double loop space is naturally a G-algebra, see Section 1; in fact, a double loop space is naturally

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an algebra over the little disks operad employed by Cohen and also Boardman and Vogt, see Section 6.2. (The reader should consult the article [30] by May in these proceedings.)

In joint work, B. Lian and G. Zuckerman (see also the joint work of M. Penkava and A. Schwarz) discovered the above-mentioned G-algebra structure in the context of topological vertex operator algebras (TVOAs) (see Section 2), which are a powerful algebraic starting point for the construction of 2D topological conformal field theories. Lian and Zuckerman also gave a number of concrete constructions of various examples of TVOAs, TCFTs and G-algebras.

In an attempt to understand Lian-Zuckerman's work geometrically, E. Getzler [14] found a G-algebra structure in the physical state space of an abstract TCFT (as well as in a topological "massive" quantum field theory). Getzler's ongoing work with J. Jones was already dealing with G-algebras as the $n = 2$ case of n-algebras. Segal's ideas, see [33], played an essential role in Getzler's discovery. In particular, Segal had already developed an extension to TCFTs of his geometric category approach [32] to conformal field theory.

Later on Y.-Z. Huang [19] found a third approach which combined the ideas of Lian, Zuckerman and Getzler. In particular, Huang took steps towards the construction of a TCFT from a TVOA; this work was based on Huang's earlier demonstration of how to construct a (tree-level) holomorphic conformal field theory (CFT) from a vertex operator algebra (VOA). Such a connection between Segal's geometric approach to conformal field theory and Borcherds' algebraic definition [6] of a VOA (see also the book of I. Frenkel, J. Lepowsky and A. Meurman [10]) had already been suggested in some public lectures by I. Frenkel [9].

From the very beginning of the above development, it was understood that the physical state space of a TCFT is merely the cohomology of a much more enormous object, the BRST complex of the TCFT. Thus there arose the question: does the G-algebra structure on the vector space of physical states come from a higher homotopy G-algebra structure on the BRST complex itself? This question was explicitly raised by Lian and Zuckerman in their work on TVOAs and associated TCFTs. They found that in the BRST complex of a TVOA, all of the identities of a G-algebra fail to hold on the nose, but they continue to hold up to homotopy. They then asked whether these homotopies could be continued to an infinite hierarchy of higher homotopies, such as those found in the work of Stasheff on A_∞-algebras and the more recent work of Stasheff and T. Lada on L_∞-algebras.

The inspiration for the search for higher homotopy algebras in topological conformal field theory arose in the related context of closed string field theory, see Stasheff [37]. The explicit discussion of higher homotopy algebras in string field theory appeared in work of Stasheff [36], M. Kontsevich [24], and E. Witten and B. Zwiebach [42, 44]. The later joint papers of T. Kimura, A. Voronov and Stasheff [22, 23] constructed L_∞ and C_∞ structures using the operadic approach. These papers also include a conceptual explanation of the relationship between string field theory and TCFTs.

However, research on higher homotopies suffered from a lack of a proper definition of a higher homotopy Gerstenhaber algebra (homotopy G-algebra). Recently, various definitions have been put forward, in particular in work of V. Ginzburg and M. Kapranov [17], Getzler and Jones [15], and Gerstenhaber and Voronov [12]. In the current paper, we use the term G_∞-algebra to refer to a particular definition of a homotopy G-algebra appearing in the work of Getzler and Jones (Definition 4.1). This definition is based on pioneering work of R. Fox and L. Neuwirth. G_∞-algebras are governed by what we call the G_∞-operad.

The main new result of the current paper is the proof that the BRST complex of a TCFT is indeed a G_∞-algebra. In fact, a (tree-level) TCFT itself is defined to be an algebra over a particular topological operad. Thus, in this paper, both the "classical" theory of topological operads as well as the recent theory of linear operads play essential roles.

The original question of Lian and Zuckerman can now be formulated precisely (see our Conjecture 2.3): does a TVOA carry a natural G_∞ structure? Since we have answered the analogous question for a TCFT, a crucial step still remains in the program to answer the original question: the completion of Huang's work on the construction of a TCFT from a TVOA. Such a construction should identify the BRST complex of the TCFT with an appropriate topological completion of the BRST complex of the TVOA. We look forward to the successful conclusion of this program.

One of the essential tools of our paper is M. Wolf and Zwiebach's "string vertices", which make a bridge between a topological operad of punctured Riemann spheres and the infinite dimensional topological operad responsible for CFTs. Amazingly, the latter operad plays a key role in the subjects of Vassiliev invariants and double loop spaces. In particular, string vertices combined with the approach of our paper yield Vassiliev invariants of knots in Section 6.1 and the structure of a homotopy G-algebra on the singular chain complex of a double loop space in Section 6.2.

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1. Gerstenhaber algebras

A *Gerstenhaber algebra* or a *G-algebra* is a graded vector space H with a dot product xy defining the structure of a graded commutative algebra and with a bracket $[x, y]$ of degree -1 defining the structure of a graded Lie algebra, such that the bracket with an element is a derivation of the dot product:

$$[x, yz] = [x, y]z + (-1)^{(\deg x-1)\deg y} y[x, z],$$

where $\deg x$ denotes the degree of an element x. In other words, a G-algebra is a specific graded version of a Poisson algebra.

This structure arises naturally in a number of contexts, such as the following.

Example 1.1. Let A be an associative algebra and $C^n(A, A) = \operatorname{Hom}(A^{\otimes n}, A)$ be its Hochschild complex. Then the dot product defined as the usual cup product up to a sign

$$(1)\quad \begin{aligned}(x \cdot y)(a_1, \dots, a_{k+l}) &= (-1)^{kl}(x \cup y)(a_1, \dots, a_{k+l})\\ &= (-1)^{kl} x(a_1, \dots, a_k) y(a_{k+1}, \dots, a_{k+l}),\end{aligned}$$

where x and y are k- and l-cochains and $a_i \in A$, and a G-bracket $[x, y]$ define the structure of a G-algebra on the Hochschild cohomology $H^n(A, A)$. The bracket was introduced by Gerstenhaber [11] in order to describe the obstruction for extending a first order deformation of the algebra A to the second order. The following definition of the bracket is due to Stasheff [35]. Considering the tensor coalgebra $T(A) = \bigoplus_{n=0}^{\infty} A^{\otimes n}$ with the comultiplication $\Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{k=0}^{n} (a_1 \otimes \cdots \otimes a_k) \otimes (a_{k+1} \otimes \cdots \otimes a_n)$, we can identify the Hochschild cochains $\operatorname{Hom}(A^{\otimes n}, A)$ with the coderivations $\operatorname{Coder} T(A)$ of the tensor coalgebra $T(A)$. Then the G-bracket $[x, y]$ is defined as the (graded) commutator of coderivations. In fact, the Hochschild complex $C^\bullet(A, A)$ is a differential graded Lie algebra with respect to this bracket.

Example 1.2. Let $A_n^\bullet$ be the $\mathbb{Z}$-graded commutative algebra generated by n variables $x_1, x_2, ..., x_n$, of degree zero, and n more variables

$\partial_{x_1}, \ldots, \partial_{x_n}$ of degree one. We refer to an element of this algebra as a polyvector field. The elements of degree zero are interpreted as functions, the elements of degree one as vector fields, those of degree two as bivector fields, and so on. The dot product is simply the graded commutative multiplication of polyvector fields.

Long ago, Schouten and Nijenhuis [31] defined a bracket operation on polyvector fields (they thought of such fields as antisymmetric contravariant tensor fields). The Schouten-Nijenhuis bracket $[P, Q]$ is characterized by the following:

1. For any two functions f and g, $[f, g] = 0$.
2. If f is a function and X is a vector field, $[X, f] = -[f, X] = Xf$.
3. If X and Y are vector fields, then $[X, Y]$ is the standard bracket of the vector fields.
4. Together, the dot product and the Schouten-Nijenhuis bracket endow $A_n^\bullet$ with the structure of a Gerstenhaber algebra.

Let C_n be the polynomial algebra in n variables. It is known that $H^\bullet(C_n, C_n)$ is canonically isomorphic as a G-algebra to the algebra $A_n^\bullet$.

Example 1.3. Let $\mathfrak{g}$ be any Lie algebra, and let $\Lambda^\bullet\mathfrak{g}$ be the Grassmann algebra generated by $\mathfrak{g}$. Define a bracket $[X, Y]$ on $\Lambda^\bullet\mathfrak{g}$ by requiring the following:

1. If a and b are scalars, $[a, b] = 0$.
2. If X is in $\mathfrak{g}$ and a is a scalar, then $[X, a] = 0$.
3. If X and Y are in $\mathfrak{g}$, then $[X, Y]$ is the Lie bracket in $\mathfrak{g}$.
4. The wedge product together with the bracket product endow $\Lambda^\bullet\mathfrak{g}$ with the structure of a Gerstenhaber algebra.

Example 1.4. Let M be a manifold (differentiable, complex, algebraic, etc.) Let $F(M)$ be the commutative algebra of functions (of the appropriate type–differentiable, holomorphic, regular, etc.) on M. Let $V^\bullet(M)$ be the algebra of polyvector fields on M, with the operations of wedge product and the Schouten-Nijenhuis bracket, defined by analogy with the bracket in $A_n^\bullet$. Then $V^\bullet(M)$ is a Gerstenhaber algebra. We can regard $V^\bullet(M)$ as the commutative superalgebra of functions on ΠT^*M, the cotangent bundle of M with the fibers made into odd supervector spaces. The G-bracket in $V^\bullet(M)$ is the odd Poisson bracket associated to the canonical odd symplectic two-form on ΠT^*M.

Let P be a bivector field on M. We can always construct a bracket operation on the algebra $F(M)$ by the formula

$$\{f, g\}_P = \iota(P)(df \wedge dg) = (df \wedge dg)(P),$$

where $\iota(P)$ denotes contraction of P against a two-form. The bracket $\{,\}_P$ satisfies the Jacobi identity if and only if the Schouten-Nijenhuis bracket $[P,P]$ is zero. In this case, the algebra $F(M)$ becomes what is known as a Poisson algebra; moreover, the derivation $\sigma_P = [P,-]$ has square zero and turns $V^\bullet(M)$ into a differential graded G-algebra, whose cohomology G-algebra is known as the Poisson cohomology of M relative to P.

Example 1.5. Let X be a topological space and let $\Omega^2 X$ be the two-fold loop space of X. Let $A^\bullet(X)$ denote the homology of $\Omega^2 X$ with rational coefficients. We endow $A^\bullet(X)$ with the structure of a $\mathbb{Z}$-graded commutative algebra, via the Pontrjagin product. $A^\bullet(X)$ is a Hopf algebra which is freely generated as a graded commutative algebra by its subspace P of primitive elements. Moreover, P is isomorphic to the rational homotopy of $\Omega^2 X$. P is $\mathbb{Z}$-graded, and we have $P_n = \pi_{n+2}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, for n nonnegative, and 0 otherwise.

The rational homotopy of the ordinary loop space of X is a $\mathbb{Z}$-graded Lie algebra, which we denote by L. The bracket is called the Samelson product. It is known that $P_n = L_{n+1}$. Thus, the algebra $A^\bullet(X)$ is isomorphic to the graded exterior algebra $\Lambda^\bullet L$. In particular, $A^\bullet(X)$ is a G-algebra of a type generalizing Example 1.3 above.

1.1. **Operads in action.** For a primer on operads, algebras over operads and the little disks operad, see P. May's paper [29] in this volume. Here we recall briefly the definition of the little disks operad in relation to G-algebras. The *little disks operad* is the collection $\{D(n),\, n \geq 1\}$ of topological spaces $D(n)$ with an action of the permutation group and operad compositions. The space $D(n)$ consists of embeddings of n little disks in the unit disk via dilatation and translation. The operad composition $\circ_i : D(m) \times D(n) \to D(m+n-1)$ is defined by contracting the unit disk with m little disks inside it to fit into the ith little disk in the other unit disk and erasing the seam. Since $D(n)$ is naturally an open subset in $\mathbb{R}^{3n}$, it is a topological operad and one can naturally obtain an operad of graded vector spaces from it, applying the functor of homology. This operad $H_\bullet(D(n))$ is called a *G-operad* in view of the following theorem.

Theorem 1.1 (F. Cohen [7, 8])**.** *The structure of a G-algebra on a $\mathbb{Z}$-graded vector space is equivalent to the structure of an algebra over the homology little disks operad $H_\bullet(D(n))$.*

2. Topological vertex operator algebras

In this section, we give a brief introduction to VOAs, in order to present the Lian-Zuckerman conjecture in an updated form. VOAs will not show up in the rest of the paper, but we intend to get back to them in a subsequent paper. See [27] for more discussion.

Definition 2.1 (Quantum operators)**.** Let $V^\bullet[\cdot] = \bigoplus_{g\in\mathbb{Z},\,\Delta\in\mathbb{Z}} V^g[\Delta]$ be an integrally bigraded complex vector space; if v is in $V^g[\Delta]$ we will write $|v| = g =$ the ghost number of v and $||v|| = \Delta =$ the weight of v. Let z be a formal variable with degrees $|z| = 0$ and $||z|| = -1$. Then, it makes sense to speak of a homogeneous *bi-infinite* formal power series

$$\phi(z) = \sum_{n\in\mathbb{Z}} \phi(n) z^{-n-1}$$

of degrees $|\phi(z)|$, $||\phi(z)||$, where the coefficients $\phi(n)$ are homogeneous linear maps in $V^\bullet[\cdot]$ of degrees $|\phi(n)| = |\phi(z)|$, $||\phi(n)|| = -n-1+||\phi(z)||$. Note then that the terms $\phi(n)z^{-n-1}$ indeed have the same degrees $|\phi(z)|$, $||\phi(z)||$ for all n. We call a finite sum of such series a *quantum operator* on $V^\bullet[\cdot]$, and denote the bigraded linear space of quantum operators as $\mathrm{QO}(V^\bullet[\cdot])$. We will denote the special operator $\phi(0)$ by the symbol $\mathrm{Res}_z\,\phi(z)$.

Definition 2.2. A *vertex operator graded algebra* consists of the following ingredients:

1. An integrally bigraded complex vector space $V^\bullet[\cdot]$.
2. A linear map $Y : V^\bullet[\cdot] \to \mathrm{QO}(V^\bullet[\cdot])$ such that Y has bidegree (0,0). We call Y the *vertex map*, and if v is in $V^\bullet[\cdot]$, we let $Y(v,z) = Y(v)(z)$ denote the *vertex operator* associated to v. The map Y is subject to the following axioms:
   (a) Let v and v' be elements of $V^\bullet[\cdot]$: then $\mathrm{Res}_z(z^m Y(v,z)v')$ vanishes for m sufficiently positive.
   (b) (Cauchy-Jacobi identity) Let v and v' be elements of $V^\bullet[\cdot]$ and let $f(z,w)$ be a Laurent polynomial in z, w, and $z-w$. Then we have the identity

$$\begin{aligned}\mathrm{Res}_w \mathrm{Res}_{z-w} Y(Y(v,z-w)v',w)f(z,w)& \\ = \mathrm{Res}_z \mathrm{Res}_w Y(v,z)Y(v',w)f(z,w)& \\ - (-1)^{|v|\,|v'|}\mathrm{Res}_w \mathrm{Res}_z Y(v',w)Y(v,z)f(z,w)&.\end{aligned}$$

   (c) There exists a distinguished element 1 in $V^0[0]$ such that $Y(1,z)$ is the identity operator and such that for any v in

$V^\bullet[\cdot]$, the result $Y(v,z)1$ is a power series in z and

$$\lim_{z\to 0} Y(v,z)1 = v.$$

3. A distinguished element F in $V^0[1]$ such that $F_0 = \operatorname{Res}_z Y(F,z)$ defines the ghost number grading: if v is in $V^g[\Delta]$, $F_0 v = gv$.
4. A distinguished element L in $V^0[2]$ such that if we define an operator $L_n = \operatorname{Res}_z(z^{n+1}Y(L,z))$ for every integer n, we have the following:
   (a) For v in $V^g[\Delta]$, $L_0 v = \Delta v$.
   (b) For any v we have $Y(L_{-1}v,z) = \partial Y(v,z)$.
   (c) For some fixed complex number c (the central charge), we have

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}1.$$

Definition 2.3. A *topological vertex operator algebra* (*TVOA*) is a vertex operator graded algebra $(V^\bullet[\cdot], Y, 1, F, L)$ equipped with two additional distinguished elements J in $V^1[1]$ and G in $V^{-1}[2]$ such that the following axioms hold:

1. The operator $Q = \operatorname{Res}_z Y(J,z)$ satisfies $Q^2 = 0$. Q is called the BRST charge, or BRST coboundary operator. Q has bidegree $(1,0)$: $|Q| = 1$, $||Q|| = 0$.
2. $[Q, Y(G,z)] = Y(L,z)$.

Definition 2.4. Let $(V^\bullet[\cdot], Y, 1, F, L, J, G)$ be a TVOA. Let v and v' be elements of $V^\bullet[\cdot]$.

1. The *dot product* of v and v' is the element

$$v \cdot v' = \operatorname{Res}_z(z^{-1}Y(v,z)v').$$

2. The *bracket product* of v and v' is the element

$$[v, v'] = (-1)^{|v|} \operatorname{Res}_w \operatorname{Res}_{z-w} Y(Y(G, z-w)v, w)v'.$$

Lemma 2.1. *Let $V^\bullet[\cdot]$ be a TVOA.*

1. *The BRST operator Q is a derivation of both the dot and bracket products, which therefore induce dot and bracket products respectively on the BRST cohomology $H^\bullet(V,Q)$ of $V^\bullet[\cdot]$ relative to Q.*
2. *Every BRST cohomology class is represented by a BRST-cocycle of weight 0. Thus, the weight grading induces the trivial grading in BRST cohomology.*
3. *The BRST cohomology is trivial unless the central charge $c = 0$.*

Theorem 2.2. *Let $V^\bullet[\cdot]$ be a TVOA. With respect to to the induced dot and bracket products, the BRST cohomology $H^\bullet(V, Q)$ of $V^\bullet[\cdot]$ relative to Q is a Gerstenhaber algebra.*

For an interesting study of the identities satisfied by the dot and bracket products in the TVOA $V^\bullet[\cdot]$ itself, see F. Akman [1]. In particular, Akman finds that the space $V^\bullet[\cdot]$ endowed with the bracket is a $\mathbb{Z}$-graded Leibniz algebra [28].

Conjecture 2.3. *Let $V^\bullet[\cdot]$ be a TVOA. Then the dot product and the skew-symmetrization of the bracket defined above can be extended to the structure of a G_∞-algebra (see below) on $V^\bullet[\cdot]$.*

This conjecture makes precise sense in the light of our current work: it refines the question posed by Lian and Zuckerman, who expected A_∞ and L_∞ structures to mix together.

3. Classical story: homotopy associative and homotopy Lie algebras

Inasmuch as G-algebras combine properties of commutative associative and Lie algebras, homotopy G-algebras make a similar combination of homotopy associative and homotopy Lie algebras; see Section 4.3 below. (The commutativity is not completely lost, either: homotopy G-algebras will also provide a homotopy for the commutativity of the dot product). Before discussing homotopy G-algebras, let us recall definitions of the more traditional homotopy associative and homotopy Lie algebras.

Definition 3.1 (Homotopy associative (A_∞-) algebras)**.** A *homotopy associative algebra* is a complex $V = \sum_{i\in\mathbb{Z}} V_i$ with a differential d, $d^2 = 0$, of degree 1 and a collection of n-ary products M_n:

$$M_n(v_1, \dots, v_n) \in V, \qquad v_1, \dots, v_n \in V,\ n \geq 2,$$

which are homogeneous of degree $2 - n$ and satisfy the relations

$$dM_n(v_1, \dots, v_n) + \sum_{i=1}^{n} \epsilon(i) M_n(v_1, \dots, dv_i, \dots, v_n)$$
$$= \sum_{\substack{k+l=n+1\\ k,l\geq 2}} \sum_{i=0}^{l-1} \epsilon(k,i) M_l(v_1, \dots, v_i, M_k(v_{i+1}, \dots, v_{i+k}), v_{i+k+1}, \dots, v_n),$$

where $\epsilon(i) = (-1)^{\deg v_1 + \dots + \deg v_{i-1}}$ is the sign picked up by taking d through $v_1, \dots, v_{i-1}$, $\epsilon(k, i) = (-1)^{k(\deg v_1 + \dots + \deg v_i)}$ is the sign picked up by M_k passing through $v_1, \dots, v_i$.

For $n = 3$, the above identity shows that the binary product M_2 is associative up to a homotopy, provided by the ternary product M_3.

Definition 3.2 (Homotopy Lie (L_∞-) algebras)**.** A *homotopy Lie algebra* is a complex $V = \sum_{i\in\mathbb{Z}} V_i$ with a differential d, $d^2 = 0$, of degree 1 and a collection of n-ary brackets:

$$[v_1, \dots, v_n] \in V, \qquad v_1, \dots, v_n \in V,\ n \geq 2,$$

which are homogeneous of degree $3 - 2n$ and super (or graded) symmetric:

$$[v_1, \dots, v_i, v_{i+1}, \dots, v_n] = (-1)^{|v_i||v_{i+1}|}[v_1, \dots, v_{i+1}, v_i, \dots, v_n],$$

$\deg v$ denoting the degree of $v \in V$, and satisfy the relations

$$d[v_1, \dots, v_n] + \sum_{i=1}^{n} \epsilon(i)[v_1, \dots, dv_i, \dots, v_n]$$
$$= \sum_{\substack{k+l=n+1\\ k,l\geq 2}} \sum_{\substack{\text{unshuffles } \sigma:\\ \{1,2,\dots,n\}=I_1\cup I_2,\\ I_1=\{i_1,\dots,i_k\},\ I_2=\{j_1,\dots,j_{l-1}\}}} \epsilon(\sigma)[[v_{i_1}, \dots, v_{i_k}], v_{j_1}, \dots, v_{j_{l-1}}],$$

where $\epsilon(i) = (-1)^{\deg v_1 + \dots + \deg v_{i-1}}$ is the sign picked up by taking d through $v_1, \dots, v_{i-1}$, $\epsilon(\sigma)$ is the sign picked up by the elements v_i passing through the v_j's during the unshuffle of $v_1, \dots, v_n$, as usual in superalgebra.

For $n = 3$, the above identity shows that the binary bracket $[v_1, v_2]$ satisfies the Jacobi identity up to a homotopy, provided by the next bracket $[v_1, v_2, v_3]$.

4. Homotopy G-algebras

Due to Fred Cohen's Theorem [8], the structure of a G-algebra on a vector space V is equivalent to the structure on V of an algebra over the homology operad $H_\bullet(D(n))$, $n \geq 1$, of the little disks operad $D(n)$. In most general terms, a homotopy Gerstenhaber algebra or homotopy G-algebra is an algebra over a "resolution" of this G-operad $G_\bullet(n) = H_\bullet(D(n))$, $n \geq 1$, i.e., an operad $hG_\bullet(n)$ of complexes whose cohomology is identified with $G_\bullet(n)$. Different resolutions lead to different notions of homotopy G-algebras. A minimal resolution to answer the question of Deligne that the Hochschild complex is a homotopy G-algebra was introduced in [12], two free resolutions were constructed in [15]. The obvious singular-chain resolution $C_\bullet(D(n))$ is too large for our purposes: it produces too many higher operations (homotopies).

Example 4.1 (Hochschild complex of an associative algebra)**.** Let A be an associative algebra and $C^n(A, A) = \mathrm{Hom}(A^{\otimes n}, A)$ its Hochschild cochain complex. Define the following collection of multilinear operations, called *braces,* on $C^\bullet(A, A)$:

$$\begin{aligned}&\{x\}\{x_1, \dots, x_n\}(a_1, \dots, a_m) := \\ &\sum (-1)^\varepsilon x(a_1, \dots, a_{i_1}, x_1(a_{i_1+1}, \dots), \dots, a_{i_n}, x_n(a_{i_n+1}, \dots), \dots, a_m)\end{aligned}$$

for $x, x_1, \dots, x_n \in C^\bullet(A, A)$, $a_1, \dots, a_m \in A$, where the summation runs over all possible substitutions of $x_1, \dots, x_n$ into x in the prescribed order and $\varepsilon := \sum_{p=1}^n (\deg x_p - 1) i_p$. The braces $\{x\}\{x_1, \dots, x_n\}$ are homogeneous of degree $-n$, i.e., $\deg\{x\}\{x_1, \dots, x_n\} = \deg x + \deg x_1 + \dots + \deg x_n - n$. We will also adopt the following convention:

$$x \circ y := \{x\}\{y\}.$$

In addition, the usual cup product (altered by a sign) defined by (1) and the differential

$$\begin{aligned}(dx)&(a_1, \dots, a_{n+1}) \\ &:= (-1)^{\deg x} a_1 x(a_2, \dots, a_{n+1}) \\ &\quad + (-1)^{\deg x} \sum_{i=1}^n (-1)^i x(a_1, \dots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \dots, a_{n+1}) \\ &\quad - x(a_1, \dots, a_n) a_{n+1}\end{aligned}$$

define the structure of a differential graded (DG) associative algebra on $C^\bullet(A, A)$. In their turn the braces satisfy the following identities.

$$\begin{aligned}(2)\quad &\{\{x\}\{x_1, \dots, x_m\}\}\{y_1, \dots, y_n\} \\ &= \sum_{0 \le i_1 \le \dots \le i_m \le n} (-1)^\varepsilon \{x\}\{y_1, \dots, y_{i_1}, \{x_1\}\{y_{i_1+1}, \dots\}, \dots, \\ &\qquad\qquad y_{i_m}, \{x_m\}\{y_{i_m+1}, \dots\}, \dots, y_n\},\end{aligned}$$

where $\varepsilon := \sum_{p=1}^m (\deg x_p - 1) \sum_{q=1}^{i_p} (\deg y_q - 1)$, i.e., the sign is picked up by the $\{x_i\}$'s passing through the $\{y_j\}$'s in the shuffle.

(3)

$$\{x_1 \cdot x_2\}\{y_1, \dots, y_n\} = \sum_{k=0}^n (-1)^\varepsilon \{x_1\}\{y_1, \dots, y_k\} \cdot \{x_2\}\{y_{k+1}, \dots, y_n\},$$

where $\varepsilon = (\deg x_2) \sum_{p=1}^{k} (\deg y_p - 1)$.

(4)

$$\begin{aligned} & d(\{x\}\{x_1, \dots, x_{n+1}\}) - \{dx\}\{x_1, \dots, x_{n+1}\} \\ & \quad - (-1)^{\deg x - 1} \sum_{i=1}^{n+1} (-1)^{\deg x_1 + \dots + \deg x_{i-1} - i - 1} \{x\}\{x_1, \dots, dx_i, \dots, x_{n+1}\} \\ = & - (-1)^{\deg x (\deg x_1 - 1)} x_1 \cdot \{x\}\{x_2, \dots, x_{n+1}\} \\ & \quad - (-1)^{\deg x} \sum_{i=1}^{n} (-1)^{\deg x_1 + \dots + \deg x_i - i} \{x\}\{x_1, \dots, x_i \cdot x_{i+1}, \dots, x_{n+1}\} \\ & \quad + (-1)^{\deg x + \deg x_1 + \dots + \deg x_n - n} \{x\}\{x_1, \dots, x_n\} \cdot x_{n+1} \end{aligned}$$

The structure of braces $\{x\}\{x_1, \dots, x_n\}$, $n \geq 1$, and a dot product xy satisfying the above identities on a complex $V = \bigoplus_n V^n$ of vector spaces was called in [12] a *homotopy G-algebra*. Thus the above braces and dot product provide the Hochschild complex $C^\bullet(A, A)$ of an associative algebra A with with the structure of a homotopy G-algebra. From the definition of a homotopy G-algebra, it is easy to find the definition of a homotopy G-operad: it is an operad of complexes with braces $\{x\}\{x_1, \dots, x_n\}$ and a dot product xy as generators and the associativity and the above identities for the braces and the dot product as defining relations.

Remark 1. A more general notion of a homotopy G-algebra is that of B_∞-algebra; see Getzler and Jones [15]. A $\mathbb{Z}$-graded vector space $V^\bullet$ is a B_∞-algebra if and only if the bar coalgebra $\mathrm{BV} = \bigoplus_{n=0}^{\infty} (V[-1])^{\otimes n}$, has the structure of a DG bialgebra, that is to say, a bialgebra with a differential which is simultaneously a derivation and a coderivation. The differential gives rise to a differential on V, as well as the dot product on V and higher A_∞ products. The multiplication, compatible with the coproduct, on BV gives rise to braces on V and some higher braces. If the higher products and the higher braces vanish identically, we retrieve the homotopy G-algebra structure of the previous example.

In this paper we will need the following still more general geometric definition of a homotopy G-algebra of Getzler and Jones [15]; see also [13]. Consider Fox-Neuwirth's cellular partition of the configuration spaces $F(n, \mathbb{C}) = \mathbb{C}^n \setminus \Delta$, where Δ is the weak diagonal $\cup_{i \neq j} \{x_i = x_j\}$, of n distinct points on the complex plane $\mathbb{C}$: cells are labeled by ordered partitions of the set $\{1, \dots, n\}$ into subsets with arbitrary reorderings within each subset. This reflects grouping points lying on common vertical lines on the plane and ordering the points lexicographically.

For each n, take the quotient cell complex $K_\bullet\mathcal{M}(n)$ by the action of translations $\mathbb{R}^2$ and dilations $\mathbb{R}^*_+$. These quotient spaces do not form an operad, but one can glue lower $K_\bullet\mathcal{M}(n)$'s to the boundaries of higher $K_\bullet\mathcal{M}(n)$'s to form a cellular operad $K_\bullet\underline{\mathcal{M}} = \{K_\bullet\underline{\mathcal{M}}(n) \mid n \geq 2\}$. In fact, the underlying spaces $\underline{\mathcal{M}}(n)$ are manifolds with corners compactifying $\mathcal{M}(n)$.

The resulting space $\underline{\mathcal{M}}(n)$ fibers over the real compactification $\underline{\mathcal{M}}_{0,n}$ of the moduli space $\mathcal{M}_{0,n}$ of n-punctured curves of genus zero, see [13, 15]. The space $\underline{\mathcal{M}}(n)$ can be also interpreted as a "decorated" moduli space, see [13]. Indeed, it can be identified with the moduli space of data $(C; p_1, \dots, p_{n+1}; \tau_1, \dots, \tau_m, \tau_\infty)$, where C is a stable algebraic curve with $n+1$ punctures $p_1, \dots, p_{n+1}$ and m double points. For each i, $1 \leq i \leq m$, τ_i is the choice of a tangent direction at the ith double point to the irreducible component that is furthest away from the "root", i.e., from the component of C containing the puncture $\infty := p_{n+1}$, while τ_∞ is a tangent direction at ∞. The stability of a curve is understood in the sense of Mumford's geometric invariant theory: each irreducible component of C must be stable, i.e., admit no infinitesimal automorphisms. The operad composition is given by attaching the ∞ puncture on a curve to one of the other punctures on another curve, remembering the tangent direction at each new double point.

Cells in this cellular operad $K_\bullet\underline{\mathcal{M}}$ are enumerated by pairs (T, p), where T is a directed rooted tree with n labeled initial vertices and one terminal vertex, labeling a component of the boundary of $\underline{\mathcal{M}}(n)$, and p is an ordered partition, as above, of the set $\mathrm{in}(v)$ of incoming edges for each vertex v of the tree T.

In [15], it is shown that a complex V is a homotopy G-algebra of the example above iff it is an algebra over the operad $K_\bullet\underline{\mathcal{M}}$ satisfying the following condition. The structure mappings

$$K_\bullet\underline{\mathcal{M}}(n) \to \mathrm{Hom}(V^{\otimes n}, V),$$

for the algebra V over the operad $K_\bullet\underline{\mathcal{M}}$ send all cells in $K_\bullet\mathcal{M}(n)$ to zero, except cells of two kinds:

1. $(\delta_n; i_1 | i_2 \dots i_n)$, where δ_n is the corolla, the tree with one root and n edges, connecting it to the remaining n vertices, corresponding to the configuration where the points $i_2, \dots, i_n$ sit on a vertical line, the i_kth point being below the i_{k+1}st, and the i_1st point is in the half-plane to the left of the line;
2. $(\delta_2; i_1 i_2)$, corresponding to the configuration where all the points sit on a single vertical line, the i_1st point being below the i_2nd.

Cells of the first kind give rise to the braces $\{x_{i_1}\}\{x_{i_2}, \dots, x_{i_n}\}$, $n \geq 2$, and cells of the second kind give rise to the dot products $x_{i_1} x_{i_2}$. The relations (3) and (4), and the associativity of the dot product, follow from the combinatorial structure of the cell complex. The relation (2) does not necessarily follow from the cell complex. Thus the homotopy G-operad of Example 4.1 is a quotient operad of $K_\bullet \underline{\mathcal{M}}$. In particular, a homotopy G-algebra of Example 2 is an algebra over $K_\bullet \underline{\mathcal{M}}$. In fact, the B_∞-operad corresponding to B_∞-algebras of Remark 1 is an intermediate operad between $K_\bullet \underline{\mathcal{M}}$ and the homotopy G-operad of Example 4.1. Thus, every B_∞-algebra is a $K_\bullet \underline{\mathcal{M}}$-algebra with certain operations also set to zero, and every homotopy G-algebra of Example 4.1 is a B_∞-algebra with some operations vanishing.

The main purpose of this paper is to give a nontrivial example of an algebra over the operad $K_\bullet \underline{\mathcal{M}}$ and thus answer the question of Lian and Zuckerman [27] about the homotopy structure of a TCFT. We will adopt the following definition of a homotopy G-operad and a homotopy G-algebra, introduced in [15] under the names of a homotopy 2-algebra and a braid algebra. We think it is appropriate to call them the G_∞-operad and a G_∞-algebra, respectively.

Definition 4.1. The G_∞*-operad* is the operad $K_\bullet \underline{\mathcal{M}}$ of complexes. An algebra over it is called a G_∞*-algebra*.

4.1. **Some lower operations.** Let V be a G_∞-algebra, as in Definition 4.1. This structure assumes a collection of n-ary operations $\mu_{T,p}$ on V associated with each cell of $K_\bullet \underline{\mathcal{M}}(n)$, i.e., with each pair (T, p), where T is an n-tree labeling a component of the boundary of $\underline{\mathcal{M}}(n)$ and p is an ordered partition of incoming vertices for each vertex of T, see above. Here we would like to illustrate this complicated algebraic structure with an explicit description of the operations and relations for $n = 2$ and 3.

For $n = 2$, $\underline{\mathcal{M}}(2) = \mathcal{M}(2) \cong S^1$ and there are just two types of cells: $(\delta_2; 1|2)$ and $(\delta_2; 12)$, dividing the circle into two intervals and two points. Denote the corresponding operations on V by $v_1 \circ v_2 = \{v_1\}\{v_2\}$ and $v_1 v_2$, where v_1 and $v_2 \in V$. The cells $(\delta_2; 2|1)$ and $(\delta_2; 21)$ are obtained by applying the transposition $\tau_{12} \in S_2$ to the above two, and therefore the corresponding operations are just $v_2 \circ v_1$ and $v_2 v_1$.

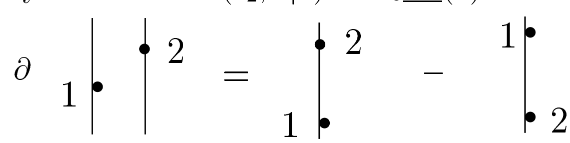
We need also to introduce the following bracket, which, unlike the circle product, induces an operation on the d-cohomology:

$$[v_1, v_2] = v_1 \circ v_2 - (-1)^{(\deg v_1 - 1)(\deg v_2 - 1)} v_2 \circ v_1. \tag{5}$$

For $n = 3$, $\mathcal{M}(3) \subset \underline{\mathcal{M}}(3)$ contains cells of the following types: $(\delta_3; 1|2|3)$, $(\delta_3; 1|23)$, $(\delta_3; 12|3)$, and $(\delta_3; 123)$, corresponding to operations which we will denote by $\{v_1\}\{v_2\}\{v_3\}$, $\{v_1\}\{v_2, v_3\}$, $\{v_1, v_2\}\{v_3\}$, and $M_3(v_1, v_2, v_3)$, respectively.

4.2. **Some lower identities.** There are no identities between compositions of operations, because the operad $\underline{\mathcal{M}}$ is free as an operad, and so is $K_\bullet \underline{\mathcal{M}}$. But there are identities involving the differential d, because the boundary of a cell in $K_\bullet \underline{\mathcal{M}}(n)$ is a linear combination of other cells — $K_\bullet \underline{\mathcal{M}}$ is a cellular operad after all. The sign rule we use for the boundaries of cells in the sequel is as follows. We introduce an orientation on cells in the configuration space $\underline{\mathcal{M}}(n)$, ordering their coordinates according to the rule: first, going from left to right, the x coordinates of the lines on which the points group, then, going lexicographically from left to right and from top to bottom, the y coordinates of the points in each group.

The boundary of the 1-cell $(\delta_2; 1|2) \in K_\bullet \underline{\mathcal{M}}(2)$ is



$$\partial(\delta_2; 1|2) = (\delta_2; 12) - (\delta_2; 21).$$

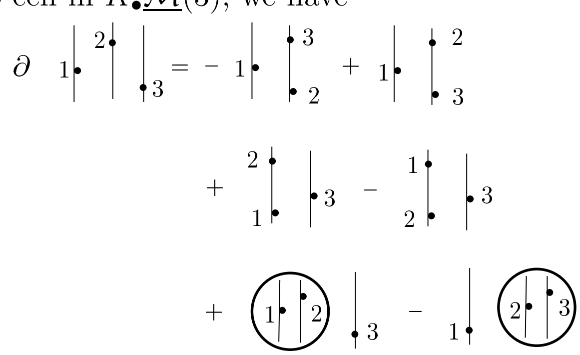
This incidence relation between cells implies the following *homotopy commutativity* relation for the dot product:

$$d(v_1 \circ v_2) - dv_1 \circ v_2 - (-1)^{\deg v_1 - 1} v_1 \circ dv_2 = v_1 v_2 - (-1)^{(\deg v_1)(\deg v_2)} v_2 v_1. \tag{6}$$

We also have $\partial(\delta_2; 12) = 0$, which yields the following *derivation property* of d with respect to the dot product:

$$d(v_1 v_2) - dv_1 v_2 - (-1)^{\deg v_1} v_1 dv_2 = 0.$$

For the top cell in $K_\bullet \underline{\mathcal{M}}(3)$, we have

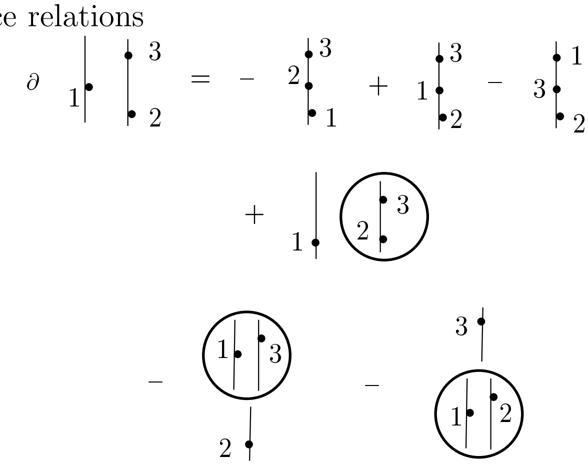


$$\begin{aligned}\partial(\delta_3; 1|2|3) = & - (\delta_3; 1|23) + (\delta_3; 1|32) \\ & + (\delta_3; 12|3) - (\delta_3; 21|3) \\ & + (\delta_2 \circ_1 \delta_2, (1|2) \circ_1 (1|2)) - (\delta_2 \circ_2 \delta_2; (1|2) \circ_2 (1|2)),\end{aligned}$$

the circles on the figure meaning "magnifying glasses", here showing that a pair of points on the sphere bubbled off onto another "microscopic" sphere attached at a double point. This happens in much the same way that a sphere pinches off to form a genus zero curve with a node as one goes to the compactification divisor in the moduli space of genus zero curves. This implies

$$\begin{aligned}(7)\quad & d(\{v_1\}\{v_2\}\{v_3\}) - \{dv_1\}\{v_2\}\{v_3\} - (-1)^{\deg v_1 - 1}\{v_1\}\{dv_2\}\{v_3\} \\ & \qquad - (-1)^{\deg v_1 + \deg v_2}\{v_1\}\{v_2\}\{dv_3\} \\ & = - \{v_1\}\{v_2, v_3\} + (-1)^{(\deg v_2 - 1)(\deg v_3 - 1)}\{v_1\}\{v_3, v_2\} \\ & \quad + \{v_1, v_2\}\{v_3\} - (-1)^{(\deg v_1 - 1)(\deg v_2 - 1)}\{v_2, v_1\}\{v_3\} \\ & \quad + (v_1 \circ v_2) \circ v_3 - v_1 \circ (v_2 \circ v_3).\end{aligned}$$

The incidence relations



$$\begin{aligned}& \partial(\delta_3; 1|23) \\ & \qquad = - (\delta_3; 123) + (\delta_3; 213) - (\delta_3; 231) \\ & \qquad\quad + (\delta_2 \circ_2 \delta_2; (1|2) \circ_2 (12)) \\ & \qquad\quad - \tau_{12}(\delta_2 \circ_2 \delta_2; (12) \circ_2 (1|2)) - (\delta_2 \circ_1 \delta_2; (12) \circ_1 (1|2))\end{aligned}$$

and similarly

$$\begin{aligned}\partial(\delta_3; 12|3)&\\ = &- (\delta_3; 123) + (\delta_3; 132) - (\delta_3; 312)\\ &- (\delta_2 \circ_1 \delta_2; (1|2) \circ_1 (12))\\ &+ (\delta_2 \circ_2 \delta_2; (12) \circ_2 (1|2)) + \tau_{12}(\delta_2 \circ_1 \delta_2; (12) \circ_1 (1|2)),\end{aligned}$$

where τ_{12} is the transposition exchanging 1 and 2, imply

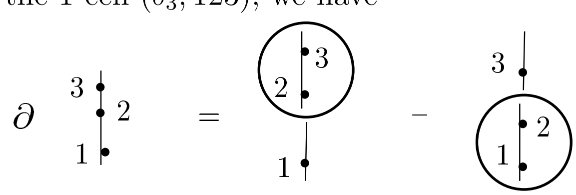
$$\begin{aligned}(8)\quad & d(\{v_1\}\{v_2, v_3\}) - \{dv_1\}\{v_2, v_3\})\\ &- (-1)^{\deg v_1 - 1}\{v_1\}\{dv_2, v_3\} - (-1)^{\deg v_1 + \deg v_2}\{v_1\}\{v_2, dv_3\}\\ &= - M_3(v_1, v_2, v_3) + (-1)^{(\deg v_1)(\deg v_2)} M_3(v_2, v_1, v_3)\\ &\quad - (-1)^{\deg v_1 (\deg v_2 + \deg v_3)} M_3(v_2, v_3, v_1) + v_1 \circ (v_2 \cdot v_3)\\ &\quad - (-1)^{(\deg v_1 - 1)\deg v_2} v_2 \cdot (v_1 \circ v_3) - (v_1 \circ v_2) \cdot v_3\end{aligned}$$

and

$$\begin{aligned}(9)\quad & d(\{v_1, v_2\}\{v_3\}) - \{dv_1, v_2\}\{v_3\})\\ &- (-1)^{\deg v_1 - 1}\{v_1, dv_2\}\{v_3\} - (-1)^{\deg v_1 + \deg v_2}\{v_1, v_2\}\{dv_3\}\\ &= - M_3(v_1, v_2, v_3) + (-1)^{(\deg v_2)(\deg v_3)} M_3(v_1, v_3, v_2)\\ &\quad - (-1)^{\deg v_3 (\deg v_1 + \deg v_2)} M_3(v_3, v_1, v_2) - (v_1 \cdot v_2) \circ v_3\\ &\quad + v_1 \cdot (v_2 \circ v_3) + (-1)^{(\deg v_3 - 1)\deg v_2} (v_1 \circ v_3) \cdot v_2,\end{aligned}$$

which can be regarded as the *homotopy left and right Leibniz rules* for the circle product with respect to the dot product, altered by the trilinear A_∞ product M_3.

Finally, for the 1-cell $(\delta_3; 123)$, we have



$$\partial(\delta_3; 123) = (\delta_2 \circ_1 \delta_2; (12) \circ_1 (12)) - (\delta_2 \circ_2 \delta_2; (12) \circ_2 (12)),$$

whence

$$\begin{aligned}(10)\quad & d(M_3(v_1, v_2, v_3)) - M_3(dv_1, v_2, v_3) - (-1)^{\deg v_1} M_3(v_1, dv_2, v_3)\\ &\qquad - (-1)^{\deg v_1 + \deg v_2} M_3(v_1, v_2, dv_3)\\ &\qquad\qquad = (v_1 v_2) v_3 - v_1 (v_2 v_3),\end{aligned}$$

so that the dot product is *homotopy associative* with M_3 being the *associator* for it. In fact, the M_n's form an A_∞-algebra, which is clearly seen at the operad level: the A_∞-operad, which is the natural cellular model of the configuration operad of the real line, embeds into $K_\bullet \underline{\mathcal{M}}$ as a real part.

Note that the commutator $[\cdot,\cdot]$ of (5) satisfies a graded *homotopy Leibniz rule*:

$$\begin{aligned}(11)\quad & [v_1, v_2v_3] - [v_1, v_2]v_3 - (-1)^{(\deg v_1 - 1)\deg v_2} v_2[v_1, v_3] \\ &= -d(\{v_1\}\{v_2, v_3\}) + \{dv_1\}\{v_2, v_3\} + (-1)^{\deg v_1 - 1}\{v_1\}\{dv_2, v_3\} \\ &+ (-1)^{\deg v_1 + \deg v_2}\{v_1\}\{v_2, dv_3\} + (-1)^{\deg v_1(\deg v_2 + \deg v_3)}(d(\{v_2, v_3\}\{v_1\} \\ &- (-1)^{\deg v_2 + \deg v_3}\{v_2, v_3\}\{dv_1\} - \{dv_2, v_3\}\{v_1\} \\ &- (-1)^{\deg v_2 - 1}\{v_2, dv_3\}\{v_1\}),\end{aligned}$$

which is a consequence of (8) and (9). The *homotopy Jacobi identity*

$$\begin{aligned}(12)\quad & [[v_1, v_2], v_3] + (-1)^{(\deg v_1 - 1)(\deg v_2 + \deg v_3)}[[v_2, v_3], v_1] \\ &+ (-1)^{(\deg v_3 - 1)(\deg v_1 + \deg v_2)}[[v_3, v_1], v_2] \\ &= \{\text{the differential of the sum of all permutations of } \{v_1\}\{v_2\}\{v_3\}\}\end{aligned}$$

for the bracket $[,]$ follows from (7). More generally, the graded symmetrizations of the operations $\{v_1\}\{v_2\}\dots\{v_n\}$ form a homotopy Lie (or L_∞-) algebra: the symmetrizations are just the fundamental cycles of $\underline{\mathcal{M}}(n)$, which freely generate an operad with the differential coming from contraction of edges of trees labeling a basis in this free operad; see Beilinson-Ginzburg [4]. Due to Hinich-Schechtman [18], this is nothing but the homotopy Lie operad. In addition, the circle product and the higher braces are a sort of universal enveloping pre-Lie algebra, cf. [11], for the L_∞-algebra.

Remark 2. The dot product v_1v_2 and the bracket $[v_1, v_2]$ descend to the cohomology of a G_∞-algebra and endow the cohomology with a G-algebra structure; see Equations (6), (10), (11), (12).

4.3. **Homotopy Gerstenhaber, associative and Lie algebras and operads.** While G-algebras combine commutative associative and Lie algebras, the G-operad describing the class of G-algebras is a natural combination of the commutative and the Lie operads $\mathcal{C}omm$ and $\mathcal{L}ie$, describing the classes of commutative and Lie algebras, respectively. Indeed, we can identify the latter operads with $H_0(\underline{\mathcal{M}}, \mathbb{C})$ and $H_{n-1}(\underline{\mathcal{M}}, \mathbb{C})$. The first identification follows from connectedness of the spaces $\underline{\mathcal{M}}(n)$: $H_0(\underline{\mathcal{M}}(n), \mathbb{C}) \cong \mathbb{C} \cong \mathcal{C}omm(n)$, the second follows Fred

Cohen's Theorem [8]: $H_{n-1}(\underline{\mathcal{M}}(n), \mathbb{C}) \cong H_{n-1}(F(n, \mathbb{C}), \mathbb{C}) \cong \mathcal{L}ie(n)$. The G-operad is just $H_\bullet(\underline{\mathcal{M}}(n), \mathbb{C})$, according to another part of Cohen's Theorem, thereby neatly interpolating the commutative and the Lie operads.

The associative operad $\mathcal{A}ssoc$ can also be seen in this moduli space picture: $H_0(\underline{\mathcal{M}}_r(n), \mathbb{C}) \cong H_0(F(n, \mathbb{R}), \mathbb{C}) \cong \mathbb{C}[S_n] \cong \mathcal{A}ssoc(n)$, where $F(n, \mathbb{R})$ is the configuration space of n points on the real line, and $\underline{\mathcal{M}}_r(n)$ is the quotient of it by translations and dilations, compactified similarly to $\underline{\mathcal{M}}(n)$ above. The natural embedding $\mathbb{R} \to \mathbb{C}$, say, as the y axis, induces a morphism $\mathcal{A}ssoc = H_0(\underline{\mathcal{M}}_r, \mathbb{C}) \to H_\bullet(\underline{\mathcal{M}}, \mathbb{C})$ of the associative operad to the G-operad, of course through the commutative one.

Similarly, as we have seen in the previous section, G_∞-algebras combine the structures of A_∞- and L_∞-algebras. Accordingly, the G_∞-operad neatly incorporates the A_∞- and the L_∞-operads. The A_∞-operad is the operad consisting of all cells of the real moduli space operad $\underline{\mathcal{M}}_r$. It embeds as a cellular operad into $K_\bullet \underline{\mathcal{M}}$ if we identify $\mathbb{R}$ with the y axis on the complex plane $\mathbb{C}$ and just consider the cells which are formed by all points grouping on a vertical line. On the other hand, the L_∞-operad is the suboperad of the G_∞-operad $K_\bullet \underline{\mathcal{M}}$ generated by the sums of all top cells of $\underline{\mathcal{M}}(n)$'s, according to Beilinson-Ginzburg-Hinich-Schechtman; see the end of the previous section.

5. Topological Field Theories

The topological field theories described here are sometimes called *cohomological field theories* and are algebras over the operad of smooth singular chains on moduli spaces of punctured Riemann spheres with decorations.

5.1. **Reduced Conformal Field Theories.** Let V be a topological vector space endowed with a collection of smooth maps $\Psi : F(n, \mathbb{C}) \to \operatorname{Hom}(V^{\otimes n}, V)$, one for each $n \geq 1$, taking $(z_1, z_2, \dots, z_n) \mapsto \Psi_{(z_1, z_2, \dots, z_n)}$ satisfying the following axioms:

1. The map Ψ should be S_n-equivariant, where the permutation group S_n acts on both $F(n, \mathbb{C})$ and $\operatorname{Hom}(V^{\otimes n}, V)$ in the obvious ways.
2. $\Psi_{(az_1+b, \dots, az_n+b)} = \Psi_{(z_1, \dots, z_n)}$ for all $a \in \mathbb{R}^+$ and $b \in \mathbb{C}$.
3. As the configuration of points $(z_1, ..., z_n)$ approaches a composition of configurations $(t_1, ..., t_p) \circ_i (w_1, ..., w_{n-p+1})$ for some $i = 1, \dots, p$

then we have

$$\Psi_{(z_1,\dots,z_n)}(v_1,\dots,v_n) \longrightarrow$$
$$\Psi_{(t_1,\dots,t_p)}(v_1,\dots,v_{i-1},\Psi_{(w_1,\dots,w_{n-p+1})}(v_i,\dots,v_{i+n-p}),v_{i+n-p+1},\dots,v_n),$$

where the compositions are understood to be in $\underline{\mathcal{M}}$. This algebraic structure is said to be a *reduced conformal field theory (reduced CFT)*.

The last axiom replaces the usual associativity axiom for VOAs and encodes the various ways in which points can come together in the plane. The operators $\Psi_{(z,0)}(v_1, v_2)$ are analogous to vertex operators $Y(v_1, z)v_2$ of a VOA. It is clear that we can redescribe this structure in the following way.

Proposition 5.1. *V is a reduced conformal field theory if and only if V is an $\underline{\mathcal{M}}$-algebra.*

A reduced topological conformal field theory (reduced TCFT) can be understood as an algebra over the smooth singular chain operad $C_\bullet(\underline{\mathcal{M}})$ of the configuration operad $\underline{\mathcal{M}}$ by analogy with TCFTs of [41]. For technical reasons — smooth singular chains, i.e., finite sums of smooth mappings from simplices to the manifold, do not naturally contain cells of $K_\bullet\underline{\mathcal{M}}$ among them — we prefer to use the following definition, imitating Segal's definition of a (full) TCFT [34], see also the next section.

Definition 5.1. A *reduced topological conformal field theory* (*reduced TCFT*) is a complex (V, d) of vector spaces, called a *BRST complex or a state space*, and a collection of operator-valued differential forms $\Omega_n \in \Omega^\bullet(\underline{\mathcal{M}}(n), \mathrm{Hom}(V^{\otimes n}, V))$, one for each $n \geq 1$, such that

1. $$\pi\Omega_n = \pi^*\Omega_n \qquad \text{for each } \pi \in S_n,$$
   where π on the left-hand side acts naturally on $\mathrm{Hom}(V^{\otimes n}, V)$ and π^* is the geometric action of π by relabeling the punctures,
2. $$d_{\mathrm{DR}}\Omega_n = d_{\mathrm{BRST}}\Omega_n,$$
   where d_{DR} is the de Rham differential and d_{BRST} is the natural differential on the space $\mathrm{Hom}(V^{\otimes n}, V)$ of n-linear operators on V,
3. $$\circ_i^*(\Omega_{m+n-1}) = \Omega_m \otimes \Omega_n \qquad \text{for each } i = 1, \dots, m,$$
   where $\circ_i : \underline{\mathcal{M}}(m) \times \underline{\mathcal{M}}(n) \to \underline{\mathcal{M}}(m + n - 1)$ is the operad law.

Theorem 5.2. *A reduced TCFT implies a natural G_∞-algebra structure on the BRST complex V.*

Proof. Given a reduced TCFT, we obtain the structure of an algebra over the G_∞-operad

$$K_\bullet \underline{\mathcal{M}}(n) \to \mathrm{Hom}(V^{\otimes n}, V), \qquad n \geq 1,$$

by integrating the forms Ω_n over the cells:

$$C \mapsto \int_C \Omega_n.$$

□

5.2. **Conformal Field Theories.** Let $\mathcal{P}(n)$ be the moduli space of Riemann spheres with $n+1$ distinct, ordered, holomorphically embedded unit disks which do not overlap except, possibly, along their boundaries. The permutation group on n elements S_n acts on $\mathcal{P}(n)$ by permuting the ordering of the first n punctures. The collection $\mathcal{P} = \{\, \mathcal{P}(n) \,\}$ forms an operad of complex manifolds where for all Σ in $\mathcal{P}(n)$ and Σ' in $\mathcal{P}(n')$, $\Sigma \circ_i \Sigma'$ in $\mathcal{P}(n+n'-1)$ is obtained by cutting out the $n'+1$st unit disk of Σ', the ith disk of Σ and sewing along their boundaries using the identification $z \mapsto \frac{1}{z}$. A *(tree level, $c=0$) conformal field theory (CFT)* is an algebra over $\mathcal{P}$. "Tree level" means that only genus 0 Riemann surfaces appear, while $c=0$ means that V is a representation of $\mathcal{P}$ rather than a projective representation (see Segal [34] for details). We shall restrict to such CFTs for simplicity. An important class of examples of holomorphic CFTs, that is, CFTs where the algebra maps are holomorphic, come from vertex operator algebras through the work of Huang-Lepowsky [20].

Definition 5.2 (Segal [34], Getzler [14])**.** A *topological conformal field theory* ($TCFT$) is a complex (V,d) and a collection of forms $\Omega_n \in \Omega^\bullet(\mathcal{P}(n), \mathrm{Hom}(V^{\otimes n}, V))$ satisfying the same axioms (1)–(3) of Definition 5.1. The complex (V,d) is sometimes called the *BRST complex or the state space* of the theory in the physics literature. Also, the cohomology of the complex is called the *space of physical states.*

Again, we shall restrict to tree level, $c=0$ TCFTs, although physical examples of TCFT's do not always have $c=0$. The forms Ω_n can be constructed using the operator formalism from algebraic data known as a *string background* (see [2, 22] for details). However, we shall not need this fact.

Proposition 5.3. *Let (V,d) be a TCFT. Then (V,d) is a reduced TCFT.*

Proof. This result follows from the fact that there exists a morphism of operads (called *string vertices*) $s : \underline{\mathcal{M}} \to \mathcal{P}$ — indeed, there exists a family of such morphisms. Consequently, any TCFT is a reduced TCFT. A similar result was used by Kimura, Stasheff, and Voronov in [22] to prove the existence of a homotopy Lie algebra structure on a natural subcomplex of a TCFT.

The construction of s uses a geometric result due to Wolf and Zwiebach [43], who showed that every conformal class of Riemann spheres with at least three punctures can be endowed with a unique metric compatible with its complex structure which solves a minimal area problem for metrics on the punctured Riemann sphere subject to the constraint that the length of any homotopically nontrivial closed curve on the punctured sphere is greater than or equal to 2π. This minimal area metric decomposes the punctured sphere into flat cylinders foliated by closed geodesics with circumference 2π. A morphism s is obtained for each pair (l, f_l) where l is a real number greater than π and $f_l : [2\pi, \infty) \to \mathbb{R}$ is a smooth monotonically increasing function such that $f_l(2\pi) = 2\pi$ and $\lim_{x\to+\infty} f_l(x) = 2l$.

Let Σ represent a point in $\underline{\mathcal{M}}(n)$; since each irreducible component of Σ can be identified with the configuration of, say m, points on $\mathbb{C}$ quotiented by dilations and translations, assign to each point a tangent direction which points along the positive real axis in $\mathbb{C}$ and then assign the minimal area metric to each irreducible component of Σ minus its punctures and double points. Shrink all internal flat cylinders in each irreducible component with a height h greater than 2π to a flat cylinder with height $f_l(h)$. Furthermore, the minimal area metric and the tangent directions at each puncture and on each side of every double point gives rise to a holomorphically embedded unit disk centered there. $s(\Sigma)$ in $\mathcal{P}(n)$ is obtained by sewing together the irreducible components on each side of every double point. The remaining curve has no remaining double points and has a holomorphically embedded unit disk around each puncture. □

Combining the previous proposition with Theorem 5.2, we conclude the following.

Corollary 5.4. *Let (V, d) be a TCFT. Then (V, d) admits the structure of a G_∞-algebra.*

The existence of an A_∞-algebra structure on the state space of a TCFT was observed in [21], cf. the homotopy associative structure of the HIKKO open string-field theory group noticed by Stasheff [36].

Remark 3. In particular, we see (Section 4.2) the structure of a homotopy commutative A_∞-algebra extending a dot product and an L_∞-algebra extending a bracket, naturally merged into one structure. This new L_∞ structure is not totally independent of the one observed by Witten and Zwiebach [42, 44] which was explained operadically in [22]. The L_∞ structure of the present paper extends the one studied before. At the operadic level, in the latter case, one projects along the phases (that is why semirelative BRST cochains are needed), while in the former one takes a section of this projection, conveniently provided by the moduli space $\underline{\mathcal{M}}$ of punctures on the sphere with an arrow at the ∞ puncture.

6. Applications of string vertices

6.1. **Vassiliev knot invariants.** Let $\widehat{\mathcal{M}}_{g,n}$ be the moduli space of stable curves of genus g and n ordered, distinct punctures whose double points are decorated with tangent directions, one on each irreducible component on either side of each double point, quotiented by the diagonal group of rotations by $U(1)$. $\widehat{\mathcal{M}}_{g,n}$ is a $6g-6+2n$ dimensional compact, oriented orbifold with corners which can be constructed by making real blowups along the irreducible components of the divisor of the moduli space of stable curves of genus g and n punctures $\overline{\mathcal{M}}_{g,n}$. The collection of spaces $\widehat{\mathcal{M}} = \{\, \widehat{\mathcal{M}}_{g,n} \,\}$ does not have natural composition maps between them, unlike $\overline{\mathcal{M}} = \{\, \overline{\mathcal{M}}_{g,n} \,\}$, because for any two punctures which are to be attached together, there is no natural way to choose tangent directions at the double points. However, the space of (smooth) singular chains $C_\bullet(\widehat{\mathcal{M}}) = \{\, C_\bullet(\widehat{\mathcal{M}}_{g,n}) \,\}$ does have natural composition maps between them by using the transfer which comes from attaching two curves together at two punctures and then averaging over the entire S^1 of tangent directions at the double points. $C_\bullet(\widehat{\mathcal{M}})$ forms a generalization of an operad called a *modular operad*, a notion due to Getzler and Kapranov [16], which generalizes operads in two ways. The first is that there is no natural "outgoing" puncture in this case — any two punctures can be attached including two on a single stable curve — while the second is that higher genus stable curves are allowed. *Throughout the remainder of this section, all operads will be assumed to be modular operads unless otherwise stated.*

$\widehat{\mathcal{M}}_{g,n}$ is a stratified space whose strata are indexed by *stable n-graphs*. These are (connected) graphs with n external legs whose vertices are decorated with a nonnegative integer. One associates a stable graph to each point in $\widehat{\mathcal{M}}_{g,n}$ by associating to each irreducible component, a corolla with an external leg associated to each puncture

and double point on that component, an integer assigned to that vertex corresponding to the genus of the irreducible component, and then attaching the corollas together whenever the irreducible components share a double point. A stratum associated to a stable n-graph consists of all points in $\widehat{\mathcal{M}}_{g,n}$ whose associated graph is the given one. The stratification of $\widehat{\mathcal{M}}_{g,n}$ is obtained by pulling up the stratification of $\overline{\mathcal{M}}_{g,n}$ via the canonical projection map $\widehat{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ which forgets the tangent directions at each double point. The union of strata gives rise to a canonical filtration of $\widehat{\mathcal{M}}$ whose associated homology spectral sequence has an E^1-term $E^1 = \{ E^1_{g,n} \}$ which forms an operad of chain complexes. E^1 contains a suboperad $\mathcal{G} = \{ \mathcal{G}_{g,n} \}$ called *the graph complex* of Kontsevich [26] which is nothing more than the Feynman transform, a generalization of the operadic bar construction to modular operads, of the commutative operad.

Kontsevich [25] showed that the homology of the graph complex $H_\bullet(\mathcal{G}_{g,n})$ has a special significance in the theory of knots. He observed that the space of primitive chord diagrams of order n is isomorphic to $\oplus_{g=0}^{n} H_0(\mathcal{G}_{g,n-g+1})_{S_{n-g+1}}$. The space of chord diagrams $\mathcal{A}$ then is nothing more than the symmetric algebra over the space of primitive chord diagrams, since $\mathcal{A}$ is a commutative, cocommutative Hopf algebra. A *weight system* is an element in $\mathcal{A}^*$. A key theorem of Kontsevich [25] and Bar-Natan [3] states that weight systems are in one to one correspondence with finite type knot invariants due to Vassiliev [39]. Such knot invariants as the Jones, Alexander-Conway polynomials and their generalizations are constructed from Vassiliev invariants. Most familiar examples of weight systems are constructed from simple Lie algebras with an invariant metric (see [3]) and their representations, although it is has recently been discovered that not all weight systems are of this kind [40]. The following theorem was very likely known to Kontsevich, who used homotopy Lie algebras with an invariant inner product and graph complexes to construct Vassiliev knot invariants.

Theorem 6.1. *Let (V, d) be an algebra over $\mathcal{G}$, Kontsevich's graph complex; then there is an associated family of weight systems.*

Proof. The algebra structure morphism $\mathcal{G} \to \mathcal{E}nd_V$ induces the morphism $m : H_\bullet(\mathcal{G}) \to \mathcal{E}nd_{H_\bullet(V)}$ which can be extended to a homomorphism of (ungraded) commutative, associative algebras $m : \mathcal{A} \to \mathcal{S}W$ where $\mathcal{S}W$ denotes the (ungraded) symmetric algebra over the vector space W. Here W is a $\mathbb{Z}$-graded vector space whose degree n subspace is

isomorphic to $\oplus_{g=0}^{n} \left(\mathcal{E}nd_{\,H_\bullet(V),(g,n-g+1)}\right)_{S_{n-g+1}}$. Therefore, given a vector in the symmetric algebra over $H_\bullet(V)$, one obtains a weight system by contracting with a suitable inner product on $H_\bullet(V)$. □

Examples of such $\mathcal{G}$-algebras come from TCFT's modulo a slight technicality. Let (V,d) be a $(c=0)$ TCFT. That is, let $\mathcal{P}_{g,n}$ be the moduli space of genus g Riemann surfaces with n holomorphically embedded unit disks which do not intersect, except possibly along the boundaries in $\mathcal{P}_{0,2}$. They assemble into the operad $\mathcal{P} = \{\,\mathcal{P}_{g,n}\,\}$. A $(c=0)$ TCFT is a collection of endomorphism-valued differential forms $\Omega_{g,n}$, $n \geq 1$, on $\mathcal{P}_{g,n}$ satisfying the natural modular operad generalization of the axioms of Definition 5.1 of Section 5.2.

Let $U(1)$ be the subgroup of $\mathcal{P}_{0,2}$ which consists of the set of Riemann spheres with the standard chart about 0 and the standard one around ∞ rotated by multiplication by a phase. If (V,d) is a TCFT then let $\Delta : V \to V$ be the unary operation associated to $U(1)$ which is regarded as a 1-cocycle in $\mathcal{P}_{0,2}$. The kernel V_{r} of Δ forms a subcomplex of (V,d) called the *semi-relative BRST complex*.

Let $\overline{\mathcal{N}}_{g,n}$ be the moduli space of stable curves of genus g with n distinct, ordered punctures which has the same decorations as points in $\widehat{\mathcal{M}}_{g,n}$ but which have, in addition, tangent directions at each puncture. The collection $\overline{\mathcal{N}} = \{\,\overline{\mathcal{N}}_{g,n}\,\}$ forms an operad. The string vertices introduced in the previous section can also be described as a morphism of (nonmodular) operads $s : \overline{\mathcal{N}} \to \mathcal{P}$ (string vertices) which are provided by minimal area metrics. These minimal area metrics are proven to exist in genus zero but are only conjectured to exist in higher genus [43]. We shall assume that they exist in what follows.

Using the string vertices, we can pull back the forms $\Omega_{g,n}$ to $\overline{\mathcal{N}}_{g,n}$ and then push them forward to forms in $\Omega^\bullet(\widehat{\mathcal{M}}_{g,n}, \mathrm{Hom}(V_{\mathrm{r}}^{\otimes n}, V_{\mathrm{r}}))$, as in [22].

Theorem 6.2. *Let (V,d) be a TCFT; then (V_{r},d) is an algebra over $\mathcal{G}$, thereby giving rise to a family of weight systems.*

Proof. By integration of the forms $\Omega_{g,n}$, (V_{r},d) becomes an algebra over Kontsevich's graph complex $\mathcal{G}$ which appears in the top row in the E^1 term in the homology spectral sequence of $\widehat{\mathcal{M}}$ associated to the canonical filtration. Now apply the previous theorem. □

It is interesting that any two given TCFT's which are homotopic through the space of TCFT's will give rise to isomorphic weight systems. Therefore, there will be families of weight systems associated to each component of the moduli space of TCFT's.

6.2. **Double loop spaces.** The following theorem, a byproduct of *string vertices*, generalizes Stasheff's characterization of loop spaces as A_∞-spaces, i.e., algebras over his polyhedra operad, to double loop spaces. It is also a refinement of the Boardman-Vogt-May-Fadell-Neuwirth characterization of double loop spaces considered up to homotopy as algebras over the little disks operad $D(n)$, $n \geq 1$.

Theorem 6.3. *Any double loop space is an algebra over the operad* $\underline{\mathcal{M}}$. *In particular, the singular chain complex of a double loop space is a homotopy G-algebra, more precisely, an algebra over the singular chain operad* $C_\bullet(\underline{\mathcal{M}})$.

Proof. The standard construction of Boardman and Vogt [5] provides a double loop space with the natural structure of an algebra over the little disks operad $D(n)$, $n \geq 1$. The same argument gives the structure of an algebra over the operad $\mathcal{P}$ of Riemann spheres with holomorphic holes. String vertices deliver an operad morphism $\underline{\mathcal{M}} \to \mathcal{P}$, which yields a morphism $C_\bullet(\underline{\mathcal{M}}) \to C_\bullet(\mathcal{P})$. □

If we found a singular chain representative of each cell in $K_\bullet(\underline{\mathcal{M}})$ compatible with the operad structure, i.e., a morphism $K_\bullet(\underline{\mathcal{M}}) \to C_\bullet(\underline{\mathcal{M}})$ of operads, we would be able to answer the following question. (In the case of Stasheff polyhedra, see more below, this morphism does exist).

Question 6.4. *Is the singular chain complex of a double loop space a* G_∞*-algebra?*

Note that string vertices offer an alternative approach to the study of loop spaces compared to that given by May's approximation theory, see for example J. Stasheff's contribution [38] to this volume. Ideally, approximation theory would provide a construction of a space $K\underline{\mathcal{M}}X$ homotopy equivalent to the double loop space $\Omega^2\Sigma^2 X$ of the double suspension of a given topological space X, such that $K\underline{\mathcal{M}}X$ is an algebra over the cellular operad $K_\bullet\underline{\mathcal{M}}$. Another approach is the content of a promised theorem of Getzler and Jones [15, Introduction].

It would be interesting to see whether string vertices exist in the case of the little intervals operad. Here the collection of string vertices should be nothing but a morphism of operads from the compactified spaces of configurations of points on the real line to the little intervals operad. Since this configuration operad is isomorphic to Stasheff's polyhedra operad, see Kontsevich [26], it may yield a simpler "quantum" proof of Stasheff's famous theorem, saying that any loop space is an algebra over the Stasheff polyhedra operad, and in particular, the

singular chain complex of a loop space is a homotopy associative (A_∞-) algebra.

References

1. F. Akman, *On some generalizations of Batalin-Vilkovisky algebras*, Preprint, Cornell University, 1995, `q-alg/9506027`.
2. L. Alvarez-Gaume, C. Gomez, G. Moore, and C. Vafa, *Strings in the operator formalism*, Nuclear Phys. B **303** (1988), 455–521.
3. D. Bar-Natan, *On Vassiliev knot invariants*, Topology **34** (1995), no. 2, 423–472.
4. A. Beilinson and V. Ginzburg, *Infinitesimal structure of moduli spaces of G-bundles*, Internat. Math. Research Notices (1992), no. 4, 63–74.
5. J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Math., vol. 347, Springer-Verlag, 1973.
6. R. E. Borcherds, *Vertex operator algebras, Kac-Moody algebras and the Monster*, Proc. Natl. Acad. Sci. USA **83** (1986), 3068–3070.
7. F. R. Cohen, *The homology of $\mathcal{C}_{n+1}$-spaces, $n \geq 0$*, The homology of iterated loop spaces, Lecture Notes in Math., vol. 533, Springer-Verlag, 1976, pp. 207–351.
8. ———, *Artin's braid groups, classical homotopy theory and sundry other curiosities*, Contemp. Math. **78** (1988), 167–206.
9. I. B. Frenkel, Lectures at the Institute for Advanced Study, January 1988.
10. I. B. Frenkel, J. Lepowsky, and A. Meurman, *Vertex operator algebras and the Monster*, Academic Press, New York, 1988.
11. M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. **78** (1963), 267–288.
12. M. Gerstenhaber and A. A. Voronov, *Higher order operations on Hochschild complex*, Functional Anal. Appl. **29** (1995), no. 1, 1–6.
13. ———, *Homotopy G-algebras and moduli space operad*, Internat. Math. Research Notices (1995), 141–153.
14. E. Getzler, *Batalin-Vilkovisky algebras and two-dimensional topological field theories*, Commun. Math. Phys. **159** (1994), 265–285, `hep-th/9212043`.
15. E. Getzler and J. D. S. Jones, *Operads, homotopy algebra and iterated integrals for double loop spaces*, Preprint, Department of Mathematics, MIT; Department of Mathematics Northwestern University, March 1994, `hep-th/9403055`.
16. E. Getzler and M. Kapranov, *Modular operads*, Preprint, Department of Mathematics, MIT, August 1994, dg-ga/9408003.
17. V. Ginzburg and M. Kapranov, *Koszul duality for operads*, Duke Math. J. **76** (1994), 203–272.
18. V. Hinich and V. Schechtman, *Homotopy Lie algebras*, Adv. Studies Sov. Math. **16** (1993), 1–18.
19. Y.-Z. Huang, *Operadic formulation of topological vertex algebras and Gerstenhaber or Batalin-Vilkovisky algebras*, Commun. Math. Phys. **164** (1994), 105–144, `hep-th/9306021`.
20. Y.-Z. Huang and J. Lepowsky, *Vertex operator algebras and operads*, The Gelfand Mathematics Seminars, 1990–1992 (Boston), Birkhäuser, 1993, `hep-th/9301009`, pp. 145–161.

21. T. Kimura, *Operads of moduli spaces and algebraic structures in topological conformal field theory*, Moonshine, the Monster, and Related Topics (Providence) (C. Dong and G. Mason, eds.), Contemporary Math., vol. 193, Amer. Math. Soc., 1996, pp. 159–190.
22. T. Kimura, J. Stasheff, and A. A. Voronov, *On operad structures of moduli spaces and string theory*, Commun. Math. Phys. **171** (1995), 1–25, `hep-th/9307114`.
23. ———, *Homology of moduli spaces of curves and commutative homotopy algebras*, The Gelfand Mathematics Seminars, 1993–1994 (J. Lepowsky and M. M. Smirnov, eds.), Birkhäuser, 1996, to appear.
24. M. Kontsevich, *Formal (non)-commutative symplectic geometry*, The Gelfand Mathematics Seminars, 1990-1992 (L. Corwin, I. Gelfand, and J. Lepowsky, eds.), Birkhäuser, 1993, pp. 173–187.
25. ———, *Vassiliev's knot invariants*, Adv. Sov. Math. **16** (1993), no. 2, 137 – 150.
26. ———, *Feynman diagrams and low-dimensional topology*, First European Congress of Mathematics, Vol. II (Paris, 1992) (Basel), Progr. Math., vol. 120, Birkhäuser, 1994, pp. 97–121.
27. B. H. Lian and G. J. Zuckerman, *New perspectives on the BRST-algebraic structure of string theory*, Commun. Math. Phys. **154** (1993), 613–646, `hep-th/9211072`.
28. J.-L. Loday, *Une version non commutative des algebres de Lie: les algebres de Leibniz*, Enseign. Math. (2) **39** (1993), 269–293.
29. J. P. May, *Definitions: Operads, algebras and modules*, Operads: Proceedings of Renaissance Conferences (J.-L. Loday, J. Stasheff, and A. A. Voronov, eds.), Amer. Math. Soc., 1996, in this volume, pp. ?–?
30. ———, *Operads, algebras and modules*, Operads: Proceedings of Renaissance Conferences (J.-L. Loday, J. Stasheff, and A. A. Voronov, eds.), Amer. Math. Soc., 1996, in this volume, pp. ?–?
31. A. Nijenhuis, *Jacobi-type identities for bilinear differential concomitants of certain tensor fields*, Indag. Math. **17** (1955), 390–403.
32. G. Segal, *Two-dimensional conformal field theories and modular functors*, IXth Int. Congr. on Mathematical Physics (Bristol; Philadelphia) (B. Simon, A. Truman, and I. M. Davies, eds.), IOP Publishing Ltd, 1989, pp. 22–37.
33. ———, Lectures at Cambridge University, summer 1992.
34. ———, *Topology from the point of view of Q.F.T.*, Lectures at Yale University, March 1993.
35. J. Stasheff, *On the homotopy associativity of H-spaces, II*, Trans. Amer. Math. Soc. **108** (1963), 293–312.
36. ———, *Higher homotopy algebras: string field theory and Drinfeld's quasi-Hopf algebras*, Proceedings of the XXth International Conference on Differential Geometric Methods in Theoretical Physics (New York, 1991) (River Edge, NJ), vol. 1, 1992, pp. 408–425.
37. ———, *Closed string field theory, strong homotopy Lie algebras and the operad actions of moduli space*, Perspectives on Mathematics and Physics (R.C. Penner and S.T. Yau, eds.), International Press, 1994, `hep-th/9304061`, pp. 265–288.
38. ———, *From operads to 'phisically' inspired theories*, Operads: Proceedings of Renaissance Conferences (J.-L. Loday, J. Stasheff, and A. A. Voronov, eds.), Amer. Math. Soc., 1996, in this volume, pp. ?–?

39. V. A. Vassiliev, *Cohomology of knot spaces*, Theory of singularities and its applications (V. I. Arnold, ed.), Amer. Math. Soc., 1990, pp. 23–69.
40. P. Vogel, *Algebraic structures on modules of diagrams*, Preprint, Université Paris VII, August 1995.
41. A. A. Voronov, *Topological field theories, string backgrounds and homotopy algebras*, Proceedings of the XXIInd International Conference on Differential Geometric Methods in Theoretical Physics, Ixtapa-Zihuatanejo, México (J. Keller and Z. Oziewicz, eds.), vol. 4, Advances in Applied Clifford Algebras (Proc. Suppl.), no. S1, 1994, pp. 167–178.
42. E. Witten and B. Zwiebach, *Algebraic structures and differential geometry in two-dimensional string theory*, Nucl. Phys. B **377** (1992), 55–112.
43. M. Wolf and B. Zwiebach, *The plumbing of minimal area surfaces*, Jour. Geom. Phys. **15** (1994), 23–56.
44. B. Zwiebach, *Closed string field theory: Quantum action and the Batalin-Vilkovisky master equation*, Nucl. Phys. B **390** (1993), 33–152.

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